

Chapter 2

2.1

- a. Let $K \equiv I[y(x)]J[y(x)]$. Then from the definition of variation and the product rule, we obtain

$$\begin{aligned}
 \delta K &= \frac{d}{d\alpha} K(y_0 + \alpha h)_{\alpha=0} = \frac{d}{d\alpha} [I(y_0 + \alpha h)J(y_0 + \alpha h)]_{\alpha=0} \\
 &= I(y_0 + \alpha h)_{\alpha=0} \frac{d}{d\alpha} [J(y_0 + \alpha h)]_{\alpha=0} + \\
 &\quad J(y_0 + \alpha h)_{\alpha=0} \frac{d}{d\alpha} [I(y_0 + \alpha h)]_{\alpha=0} \\
 &= I(y_0) \frac{d}{d\alpha} [J(y_0 + \alpha h)]_{\alpha=0} + J(y_0) \frac{d}{d\alpha} [I(y_0 + \alpha h)]_{\alpha=0} \\
 &= I(y_0) \delta J(y_0; h) + J(y_0) \delta I(y_0; h)
 \end{aligned}$$

- b. Let $K \equiv \frac{I[y(x)]}{J[y(x)]}$ where $J[y(x)] \neq 0$. Then from the definition of variation and the product rule, we obtain

$$\begin{aligned}
 \delta K &= \frac{d}{d\alpha} K(y_0 + \alpha h)_{\alpha=0} = \frac{d}{d\alpha} \left[\frac{I(y_0 + \alpha h)}{J(y_0 + \alpha h)} \right]_{\alpha=0} \\
 &= I(y_0 + \alpha h)_{\alpha=0} \frac{d}{d\alpha} \left[\frac{1}{J(y_0 + \alpha h)} \right]_{\alpha=0} + \\
 &\quad \frac{1}{J(y_0 + \alpha h)}_{\alpha=0} \frac{d}{d\alpha} [I(y_0 + \alpha h)]_{\alpha=0} \\
 &= -\frac{I(y_0)}{[J(y_0)]^2} \frac{d}{d\alpha} [J(y_0 + \alpha h)]_{\alpha=0} + \frac{1}{J(y_0)} \frac{d}{d\alpha} [I(y_0 + \alpha h)]_{\alpha=0} \\
 &= -\frac{I(y_0)}{[J(y_0)]^2} \delta J(y_0; h) + \frac{\delta I(y_0; h)}{J(y_0)}
 \end{aligned}$$

- 2.2 Substituting Equation (1.11) into Equation (1.12), we get

$$I = y_0 + \tau S \int_0^L \left[\frac{-k_1 y P}{2y_0 - y} + \frac{4k_2 (y_0 - y)^2 P^2}{(2y_0 - y)^2} \right] dz$$

Then the variation of I is given by

$$\begin{aligned} \delta I = \tau S \int_0^L \left\{ \left[-\frac{k_1 P}{2y_0 - y} - \frac{k_1 y P}{(2y_0 - y)^2} \right] \delta y - \left[\frac{k_1 y}{2y_0 - y} \right] \delta P + \right. \\ \left[-\frac{8k_2(y_0 - y)P^2}{(2y_0 - y)^2} + \frac{8k_2(y_0 - y)^2 P^2}{(2y_0 - y)^3} \right] \delta y + \\ \left. \left[\frac{8k_2(y_0 - y)^2 P}{(2y_0 - y)^2} \right] \delta P \right\} dz \end{aligned}$$

2.3 The augmented functional is given by

$$\begin{aligned} J = I + \int_0^\tau \left\{ \lambda_1 \left[-\dot{x} + \frac{F_x x_f}{V} - \frac{(F_x + F_y)x}{V} - k_x xy \right] + \lambda_2 \left[-\dot{y} + \frac{F_y y_f}{V} - \right. \right. \\ \left. \frac{(F_x + F_y)y}{V} - k_x xy - k_p y \mu_0 \right] + \lambda_3 \left[-\dot{\mu}_0 + k_x xy - \frac{(F_x + F_y)\mu_0}{V} \right] \\ + \lambda_4 \left[-\dot{\mu}_1 + k_x xy - \frac{(F_x + F_y)\mu_1}{V} + k_p y \mu_0 \right] + \lambda_5 \left[-\dot{\mu}_2 + k_x xy - \right. \\ \left. \frac{(F_x + F_y)\mu_2}{V} + k_p y (\mu_0 + 2\mu_1) \right] \left. \right\} dt \end{aligned}$$

where λ_i s are the Lagrange multipliers. The variation of J is then given by

$$\begin{aligned} \delta J = 2 \left[\frac{1}{\tau} \int_0^\tau \frac{\mu_0 \mu_2}{\mu_1^2} dt - D^* \right] \frac{1}{\tau} \int_0^\tau \left(\frac{\mu_2}{\mu_1^2} \delta \mu_0 - \frac{2\mu_0 \mu_2}{\mu_1^3} \delta \mu_1 + \frac{\mu_0}{\mu_1^2} \delta \mu_2 \right) dt + \\ \int_0^\tau \left\{ \lambda_1 \left[-\delta \dot{x} + \frac{x_f}{V} \delta F_x - \frac{x}{V} (\delta F_x + \delta F_y) - \frac{(F_x + F_y)}{V} \delta x - k_x (x \delta y + y \delta x) \right] \right. \\ + \lambda_2 \left[-\delta \dot{y} + \frac{y_f}{V} \delta F_y - \frac{y}{V} (\delta F_x + \delta F_y) - \frac{(F_x + F_y)}{V} \delta y - k_x (x \delta y + y \delta x) \right. \\ \left. - k_p (\mu_0 \delta y + y \delta \mu_0) \right] + \lambda_3 \left[-\delta \dot{\mu}_0 + k_x (x \delta y + y \delta x) - \frac{\mu_0}{V} (\delta F_x + \delta F_y) \right. \\ \left. - \frac{(F_x + F_y)}{V} \delta \mu_0 \right] + \lambda_4 \left[-\delta \dot{\mu}_1 + k_x (x \delta y + y \delta x) - \frac{\mu_1}{V} (\delta F_x + \delta F_y) \right. \end{aligned}$$

$$\begin{aligned}
& -\frac{(F_x + F_y)}{V}\delta\mu_1 + k_p(\mu_0\delta y + y\delta\mu_0) \Big] + \lambda_5 \Big[-\delta\dot{\mu}_2 + k_x(x\delta y + y\delta x) \\
& -\frac{\mu_2}{V}(\delta F_x + \delta F_y) + k_p(\mu_0 + 2\mu_1)\delta y + k_p y(\delta\mu_0 + 2\delta\mu_1) \Big] \Big\} dt
\end{aligned}$$

2.4 The augmented functional is given by

$$J = \int_0^{t_f} \left[u^2 + \lambda_1 \left(-\dot{x} + \frac{rx - \alpha x^2 - \beta x}{\tau} + \mu y \right) + \lambda_2 \left(-\dot{y} + \frac{\beta x}{\tau} - \mu y \right) \right] dt$$

where λ_i s are the Lagrange multipliers. The variation of J is then given by

$$\begin{aligned}
\delta J = \int_0^{t_f} & \left[2u\delta u + \lambda_1 \left(-\delta\dot{x} + \frac{x r_u}{\tau}\delta u + \frac{r - 2\alpha x - \beta}{\tau}\delta x + \mu\delta y \right) + \right. \\
& \left. \lambda_2 \left(-\delta\dot{y} + \frac{\beta}{\tau}\delta x - \mu\delta y \right) \right] dt
\end{aligned}$$

2.5 Expand each term involving the variation of a derivative using integration by parts. For example,

$$\int_0^\tau \lambda_1 \delta \dot{x} dt = [\lambda_1 \delta x]_0^\tau - \int_0^\tau \dot{\lambda}_1 \delta x dt$$

Applying the above expansion to Exercise 2.3, we obtain

$$\begin{aligned}
\delta J = 2 & \left[\frac{1}{\tau} \int_0^\tau \frac{\mu_0 \mu_2}{\mu_1^2} dt - D^* \right] \frac{1}{\tau} \int_0^\tau \left(\frac{\mu_2}{\mu_1^2} \delta\mu_0 - \frac{2\mu_0 \mu_2}{\mu_1^3} \delta\mu_1 + \frac{\mu_0}{\mu_1^2} \delta\mu_2 \right) dt + \\
& \int_0^\tau \left\{ \lambda_1 \left[\frac{x_f}{V} \delta F_x - \frac{x}{V} (\delta F_x + \delta F_y) - \frac{(F_x + F_y)}{V} \delta x - k_x(x\delta y + y\delta x) \right] \right. \\
& + \lambda_2 \left[\frac{y_f}{V} \delta F_y - \frac{y}{V} (\delta F_x + \delta F_y) - \frac{(F_x + F_y)}{V} \delta y - k_x(x\delta y + y\delta x) \right. \\
& \left. \left. - k_p(\mu_0\delta y + y\delta\mu_0) \right] + \lambda_3 \left[k_x(x\delta y + y\delta x) - \frac{\mu_0}{V} (\delta F_x + \delta F_y) \right] \right\} dt
\end{aligned}$$

$$\begin{aligned}
& -\frac{(F_x + F_y)}{V}\delta\mu_0 \Big] + \lambda_4 \left[k_x(x\delta y + y\delta x) - \frac{\mu_1}{V}(\delta F_x + \delta F_y) \right. \\
& -\frac{(F_x + F_y)}{V}\delta\mu_1 + k_p(\mu_0\delta y + y\delta\mu_0) \Big] + \lambda_5 \left[k_x(x\delta y + y\delta x) \right. \\
& \left. -\frac{\mu_2}{V}(\delta F_x + \delta F_y) + k_p(\mu_0 + 2\mu_1)\delta y + k_p y(\delta\mu_0 + 2\delta\mu_1) \right] \Big\} dt \\
& + \int_0^\tau \left(\dot{\lambda}_1\delta x + \dot{\lambda}_2\delta y + \dot{\lambda}_3\delta\mu_0 + \dot{\lambda}_4\delta\mu_1 + \dot{\lambda}_5\delta\mu_2 \right) dt \\
& - \underbrace{\left[\lambda_1\delta x + \lambda_2\delta y + \lambda_3\delta\mu_0 + \lambda_4\delta\mu_1 + \lambda_5\delta\mu_2 \right]_0^\tau}_{A=0}
\end{aligned}$$

where the last group of terms denoted by A is zero due to periodic conditions. Thus, if $x(0) = x(\tau)$, then $\delta x(0) = \delta x(\tau)$ and the corresponding terms in A vanish.

In a similar manner, we obtain for Exercise 2.4,

$$\begin{aligned}
\delta J = \int_0^{t_f} \left[2u\delta u + \lambda_1 \left(\frac{x r_u}{\tau} \delta u + \frac{r - 2\alpha x - \beta}{\tau} \delta x + \mu \delta y \right) + \right. \\
\left. \lambda_2 \left(\frac{\beta}{\tau} \delta x - \mu \delta y \right) \right] dt + \int_0^{t_f} \left(\dot{\lambda}_1 \delta x + \dot{\lambda}_2 \delta y \right) dt - \left[\lambda_1 \delta x + \lambda_2 \delta y \right]^\tau
\end{aligned}$$

where we have utilized the fact that $\delta x(0) = \delta y(0) = 0$ since $x(0)$ and $y(0)$ are specified, i.e., fixed.

2.6 The augmented objective functional is given by

$$\begin{aligned}
J = \int_0^\tau \left[1 - \frac{T}{T_e} \right]^2 dt + \int_0^\tau \int_{r_0}^R \lambda \left[-\dot{T} + \frac{k}{r\rho C_p} (T_r + rT_{rr}) + \right. \\
\left. \frac{FC_b(T_b - T)}{C_p} + \frac{\Delta H}{\rho C_p} \right] dr dt
\end{aligned}$$

where the Lagrange multiplier λ depends on t as well as r . The variation of J is given by

$$\delta J = -2 \int_0^\tau \left(1 - \frac{T}{T_e}\right) \delta T \, dt + \int_0^\tau \int_{r_0}^R \lambda \left[-\delta \dot{T} + \frac{k}{r \rho C_p} (\delta T_r + r \delta T_{rr}) + \frac{C_b}{C_p} [(T_b - T) \delta F - F \delta T] + \frac{\delta(\Delta H)}{\rho C_p} \right] dr \, dt$$

Using integration by parts,

$$\begin{aligned} \int_0^\tau \int_{r_0}^R \lambda \delta \dot{T} \, dr \, dt &= \int_{r_0}^R \int_0^\tau \lambda \delta \dot{T} \, dt \, dr = \int_{r_0}^R \underbrace{\left[\lambda \delta T \right]_0^\tau}_{\text{zero due to periodicity}} \, dz - \int_{r_0}^R \int_0^\tau \dot{\lambda} \delta T \, dt \, dz \\ &= - \int_0^\tau \int_{r_0}^R \dot{\lambda} \delta T \, dr \, dt \\ \int_0^\tau \int_{r_0}^R \frac{k \lambda}{r \rho C_p} \delta T_r \, dr \, dt &= \int_0^\tau \left[\frac{k \lambda}{r \rho C_p} \delta T \right]_{r_0}^R \, dt - \int_0^\tau \int_{r_0}^R \left[\frac{k \lambda_r}{r \rho C_p} - \frac{k \lambda}{r^2 \rho C_p} \right] \delta T \, dr \, dt \\ \int_0^\tau \int_{r_0}^R \frac{k \lambda}{\rho C_p} T_{rr} \, dr \, dt &= \int_0^\tau \left[\frac{k \lambda}{\rho C_p} \delta T_r \right]_{r_0}^R \, dt - \int_0^\tau \int_{r_0}^R \frac{k \lambda_r}{\rho C_p} \delta T_r \, dr \, dt \\ &= \int_0^\tau \left[\frac{k \lambda}{\rho C_p} \delta T_r \right]_{r_0}^R \, dt - \int_0^\tau \left[\frac{k \lambda_r}{\rho C_p} \delta T \right]_{r_0}^R \, dt + \int_0^\tau \int_{r_0}^R \frac{k \lambda_{rr}}{\rho C_p} \delta T \, dr \, dt \end{aligned}$$

Substituting the last three equations into the expression for δJ , we get

$$\begin{aligned} \delta J &= -2 \int_0^\tau \left(1 - \frac{T}{T_e}\right) \delta T \, dt + \int_0^\tau \int_{r_0}^R \lambda \left\{ \frac{C_b}{C_p} [(T_b - T) \delta F - F \delta T] + \frac{\delta(\Delta H)}{\rho C_p} \right\} dr \, dt \\ &\quad + \int_0^\tau \left[\frac{k}{r \rho C_p} \delta T + \frac{k \lambda}{\rho C_p} \delta T_r - \frac{k \lambda_r}{\rho C_p} \delta T \right]_{r_0}^R \, dt + \int_0^\tau \int_{r_0}^R \left[\dot{\lambda} + \frac{k \lambda_{rr}}{\rho C_p} \right] \delta T \, dr \, dt \\ &\quad - \int_0^\tau \int_{r_0}^R \left[\frac{\lambda_r k}{r \rho C_p} - \frac{k \lambda}{r^2 \rho C_p} \right] \delta T \, dr \, dt \end{aligned}$$

2.7 The augmented functional is given by

$$J = \int_0^{t_f} [-Dc_x(L) - J^*]^2 dt + \int_0^{t_f} \int_0^L \lambda(-\dot{c} + Dc_{xx}) dx dt$$

where λ is the Lagrange multiplier, which depends on x and t . The variation of J is given by

$$\delta J = -2 \int_0^{t_f} [Dc_x(L) + J^*] D\delta c_x(L) dt + \int_0^{t_f} \int_0^L \lambda(-\delta\dot{c} + D\delta c_{xx}) dx dt$$

Using integration by parts, we get

$$\begin{aligned} \int_0^{t_f} \int_0^L \lambda \delta \dot{c} dx dt &= \int_0^{t_f} \int_0^L \lambda \delta \dot{c} dt dx = \int_0^{t_f} [\lambda \delta c]_0^L dx - \int_0^{t_f} \int_0^L \dot{\lambda} \delta c dx dt \\ &= \int_0^{t_f} [\lambda \delta c]_0^L dx - \int_0^{t_f} \int_0^L \dot{\lambda} \delta c dx dt \\ \int_0^{t_f} \int_0^L \lambda D \delta c_{xx} dx dt &= \int_0^{t_f} [\lambda D \delta c_x]_0^L dt - \int_0^{t_f} \int_0^L D \lambda_x \delta c_x dx dt \\ &= \int_0^{t_f} [\lambda D \delta c_x]_0^L dt - \int_0^{t_f} [D \lambda_x \delta c]_0^L dt + \int_0^{t_f} \int_0^L (D \lambda_{xx} \delta c) dx dt \end{aligned}$$

Substituting the above results in the expression of δJ , we get

$$\begin{aligned} \delta J &= -2 \int_0^{t_f} [Dc_x(L) + J^*] D\delta c_x(L) dt - \int_0^{t_f} [\lambda \delta c]_0^L dx \\ &\quad + \int_0^{t_f} [\lambda D \delta c_x - D \lambda_x \delta c]_0^L dt + \int_0^{t_f} \int_0^L (\dot{\lambda} + D \lambda_{xx}) \delta c dx dt \end{aligned}$$

where the terms involving δc at $x = L$ are zero since $c(L, t)$ is specified, i. e., fixed.

2.8 All the involved partial derivatives must be continuous (see Section 2.5.1, p. 41).

2.9 If $y_0 < 0$, then we can always find a small enough $\epsilon > 0$ for which $y_0 + \epsilon < 0$. Let $y_0 = -a^2$ and $y_0 + \Delta y = -b^2$ lie in the interval $[y_0 - \epsilon, y_0 + \epsilon]$. Let Δy be positive and equal to c^2 . Then $b^2 - a^2 = -c^2$ and

$$\begin{aligned} \left[\frac{d}{dy} |y| \right]_{\alpha=0} &= \lim_{\Delta y \rightarrow 0} \left[\frac{|y_0 + \Delta y| - |y_0|}{\Delta y} \right] = \lim_{\Delta y \rightarrow 0} \left[\frac{|-b^2| - |-a^2|}{\Delta y} \right] \\ &= \lim_{\Delta y \rightarrow 0} \left[\frac{-c^2}{c^2} \right] = -1 \end{aligned}$$

Thus

$$dI(y_0; h) = 2 \int_0^1 \left[\frac{d}{dy} |y| \right]_{\alpha=0} x \, dx = -1$$

We get the same result if Δy is negative and equal to $-c^2$.

Likewise for $y_0 > 0$, it can be easily shown that $dI(y_0; h) = 1$.

Chapter 3

3.1 The Hamiltonian is given by

$$H = u^2 - y_2 + \lambda_1(-y_1 u + y_1 u^2) + \lambda_2(y_2 u)$$

The necessary conditions for the minimum are

$$\begin{aligned} \dot{y}_1 &= H_{\lambda_1} = -y_1 u + y_1 u^2, & y_1(0) &= y_{1,0} \\ \dot{y}_2 &= H_{\lambda_2} = y_2 u, & y_2(0) &= 0 \\ \dot{\lambda}_1 &= -H_{y_1} = \lambda_1(u - u^2), & \lambda_1(t_f) &= 0 \\ \dot{\lambda}_2 &= -H_{y_2} = 1 - \lambda_2 u, & \lambda_2(t_f) &= 0 \\ H_u &= 2u + \lambda_1(-y_1 + 2y_1 u) + \lambda_2 y_2 = 0 \end{aligned}$$

3.2 Equate δI to zero in Exercise 2.2. Then the necessary conditions for the minimum are the coefficients of δy and δP equated to zero and the initial state. Alternatively, discard the additive constant y_0 appearing in the expressing for I and utilize the Hamiltonian

$$\begin{aligned} H &= \frac{dy}{dz} + \lambda \tau S \left[\frac{-k_1 y P}{2y_0 - y} + \frac{4k_2(y_0 - y)^2 P^2}{(2y_0 - y)^2} \right] \\ &= (1 + \lambda) \tau S \left[\frac{-k_1 y P}{2y_0 - y} + \frac{4k_2(y_0 - y)^2 P^2}{(2y_0 - y)^2} \right] \end{aligned}$$

to obtain the following necessary conditions for the minimum:

$$\frac{dy}{dz} = H_\lambda = \tau S \left[\frac{-k_1 y P}{2y_0 - y} + \frac{4k_2(y_0 - y)^2 P^2}{(2y_0 - y)^2} \right], \quad y(0) = y_0$$

$$\frac{d\lambda}{dz} = -H_y = -(1 + \lambda) \tau S \left[\frac{-k_1 P}{2y_0 - y} - \frac{k_1 y P}{(2y_0 - y)^2} - \frac{8k_2(y_0 - y) P^2}{(2y_0 - y)^2} + \frac{8k_2(y_0 - y)^2 P^2}{(2y_0 - y)^3} \right], \quad \lambda(t_f) = 0$$

$$H_P = \frac{-k_1 y}{2y_0 - y} + \frac{8k_2(y_0 - y)^2 P}{(2y_0 - y)^2} = 0$$

3.3 This exercise is similar to Exercise 3.1 except for a greater number of state equations and control functions.

3.4 Introduce $y_0 \equiv t$, which means that

$$\dot{y}_0 = 1 \quad \text{and} \quad \dot{y} = g(y_0, y, u)$$

are the two state equations that govern I . In terms of

$$\mathbf{y} \equiv [y_0 \quad y]^\top \quad \text{and} \quad \mathbf{g} \equiv [1 \quad g]^\top$$

the problem is autonomous with the objective to find the minimum of

$$I = \int_0^{t_f} F[\mathbf{y}(t), u(t)] dt$$

subject to $\dot{\mathbf{y}} = \mathbf{g}$.

Chapter 4

4.1

- a. According to the preconditions for the optimum (see p.102), the partial derivatives of F and \mathbf{g} are continuous with respect to \mathbf{y} and \mathbf{u} . Then from the existence theorem (Cauchy–Peano theorem), the solutions of

$$\dot{\mathbf{y}} = \mathbf{g}, \quad \mathbf{y}(0) = \mathbf{y}_0$$

$$\dot{\lambda} = -H_{\mathbf{y}} = -F_{\mathbf{y}} - \lambda^\top \mathbf{g}_{\mathbf{y}}, \quad \lambda(t_f) = \mathbf{0}$$

exist and are continuous with respect to time.